Reaction-diffusion models describing a two-lane traffic flow

M. Ebrahim Fouladvand*
Department of Physics, Sharif University of Technology, P.O. Box 11365-9161, Tehran, Iran
and Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5531, Tehran, Iran
(Received 24 March 2000)

A unidirectional two-lane road is approximated by a set of two parallel closed one-dimensional chains. Two types of cars, i.e., slow and fast ones are considered in the system. Based on the Nagel-Schreckenberg model of traffic flow [K. Nagel and M. Schreckenberg, J. Phys. **2**, 2221 (1992)], a set of reaction-diffusion processes is introduced to simulate the behavior of the cars. Fast cars can pass the slow ones using the passing lane. We write and solve the mean-field rate equations for the density of slow and fast cars, respectively. We also investigate the properties of the model through computer simulations and obtain the fundamental diagrams. A comparison between our results and the $v_{max}=2$ version of the Nagel-Schreckenberg model is made.

PACS number(s): 02.50.Ey, 05.70.Ln, 05.70.Fh, 82.20.Mj

I. INTRODUCTION

In recent years, modeling traffic flow has been the subject of comprehensive studies by statistical physicists [1–5]. Needless to say, many general phenomena in vehicular traffic can be explained in general terms with these models. Distinct traffic states have been identified and some of these models have found empirical applications in real traffic [2–4]. In these investigations, various theoretical approaches, namely, microscopic car-following models [6,7], hydrodynamical coarse-grained macroscopic models [5,8,9], and gas-kinetic models [10,11], have been developed in order to find a better quantitative as well as qualitative understanding toward vehicular traffic phenomena. Recently, as an alternative microscopic description, probabilistic cellular automata (PCA) have come into play (for an overview, see Refs. [12,13]). This approach to theoretical description of traffic flow is one of the most effective and well-established ones and there is a relatively rich amount of results, both numeric and analytic, in the literature [1,14].

In PCA models, space (road), time, and velocities of vehicles are assumed to take discrete values. This realization of traffic flow makes PCA an ideal tool for the computer simulation. One of the prototype PCA models is the so-called Nagel-Schreckenberg (NS) model [15], which describes a single-lane traffic flow. Although the initial observations of the NS model were numerical, shortly thereafter, analytical techniques were also proposed [12–14]. Analytical treatments to CA are difficult in general. This is mainly due to the discreteness and the use of parallel (synchronous) updating procedures that produce the largest correlation among the vehicles with regard to other updating schemes. Soon after its introduction, the NS model was extended to account for more realistic situations such as multilane traffic flow [16,17], bidirectional roads [18], and urban traffic [19,20]. In multilane traffic, fast cars are capable of passing the slow ones by using the fast lane. The possibility of lane changing allows for these models to exhibit nontrivial and interesting properties that are exclusive to multilane traffic flow. Despite the quite large approximative methods applied to single-lane NS based models, there are few analytical approaches to multilane traffic flow [21]. One main reason is the large number of rules in PCA modeling of multilane traffic. In reality, a driver attempting to overtake the car ahead (in a unidirectional road) has to take the following criteria into consideration.

(1) There must be enough forward space in the passing lane.

(2) There must be enough backward space in the passing lane so that no accident could occur between two simultaneously passing cars.

Moreover, in bidirectional roads, additional criteria are necessary for a successful passing (for details see [18]). The main purpose of the present paper is to introduce an analytical approach to study a unidirectional two-lane road. The approach we use is to some extent similar to PCA, however, basic differences are distinguishable. The major distinction is concerned with the type of updating scheme. In contrast to PCA, which are realized in parallel update, our models are based on time-continuous random sequential update. The mechanism of modeling the two-lane traffic we use is based on the stochastic reaction-diffusion processes, however, the rules have roots in the NS rules. This paper is organized as follows: In Sec. II, we define the first model (model I) and interpret the rules in terms of those in the NS model. Section III starts with the Hamiltonian description of the related master equation and continues with mean-field rate equations and their solutions. The results of the numerical simulation of model I ends this section. Next, we introduce the model II in Sec. IV, which is formulated in symmetric as well as asymmetric versions and follow the same steps performed in Sec. III to obtain the fundamental diagrams of the both versions. The paper ends with some concluding remarks in Sec. V.

II. DEFINITIONS OF THE MODELS

In the first model, a unidirectional two-lane road is approximated by a set of two parallel one-dimensional chains, each with N sites. The periodic boundary condition applies to both. Cars are considered as particles that occupy sites of the chains. Two types of cars exist in the system: slow cars, which are denoted by A, and fast cars, denoted by B. Also, Φ

*Email address: foolad@theory.ipm.ac.ir

1063-651X/2000/62(5)/5940(8)/$15.00

represents an empty site. Each site of the chains is either empty, occupied by a slow or by a fast car.

Fast cars can pass the slow ones with certain probabilities while approaching them. The bottom lane is the home lane and cars are only allowed to use the top lane for passing. Once the passing process is achieved, they should return to the home lane. This realization of a two-lane road is regarded as ''asymmetric'' type. Nonetheless ''symmetric'' type could also be implemented where passing from the right is allowed as well. In model I, we restrict ourselves to ''asymmetric'' type. The state of the system is characterized by two sets of occupation numbers $(\xi_1,\xi_2,\ldots,\xi_N)$ and $(\sigma_1,\sigma_2,\ldots,\sigma_N)$ for the home and passing lane, respectively. $\xi_i,\sigma_i=0,1,2$, where zero refers to an empty site whereas one and two refer to a site being occupied by a slow or a fast car, respectively.

To investigate the characteristics of this model, a simplification has been considered. If simultaneous two-car occupation of parallel sites of the chains is forbidden, one can describe configurations with a single set of occupation numbers $\{\tau_i\}$, where $\tau_i=0,1,2$.

Inspired by the $v_{max}=2$ version of the NS model [15,12], we propose the following set of stochastic processes that evolve according to a random sequential updating scheme:

$$A\Phi\rightarrow\Phi A \quad (\text{with rate } h), \tag{1}$$

$$B\Phi\rightarrow\Phi B \quad (\text{with rate } p), \tag{2}$$

$$A\Phi\rightarrow\Phi B \quad (\text{with rate } q), \tag{3}$$

$$B\Phi\rightarrow\Phi A \quad (\text{with rate } r), \tag{4}$$

$$BA\rightarrow AA \quad (\text{with rate } \lambda), \tag{5}$$

$$BA\Phi\rightarrow\Phi AB \quad (\text{with rate } s). \tag{6}$$

In order to illustrate the above definitions, let us express their interpretations.

The first and the second of the above rules correspond to the free moving of slow and fast cars, respectively. The third one expresses the accelerated movement of slow cars. This step corresponds to the so-called acceleration step in the NS model. The fourth rule simulates the behavior of a driver randomly reducing his/her speed as a result of environmental effects, road conditions, etc. This step corresponds to the so-called ''random breaking'' step in the NS model. Finally, the last two processes simulate the behavior of the fast-car drivers when approaching a slow car. Either they pass the slow car using the passing lane or they prefer to move behind it, which give rises to their speed reduction.

We recall that in the NS model, the forward movement of each car is highly affected by the car ahead. Here, for simplicity, we have considered the two-site interactions and only use three-site interaction for the passing process. In this particular case, it is crucial that the site ahead of the slow car should be empty. Despite the partial explanation of microscopic rules necessary for the description of a traffic flow in a two-lane road, the present model ignores the effect of oncoming fast cars (in the passing lane) on the fast car (in the home lane). In reality, a fast car attempts to overtake provided that there is enough back-space behind him in the passing lane, i.e., there is no passing car close to him in the passing lane [16,17]. In model I, passing occurs locally and irrespective of the state of passing lane behind the fast car in the home lane.

III. MASTER EQUATION AND MEAN-FIELD RATE EQUATIONS

The processes (1)–(6) could be regarded as a two-species, one-dimensional reaction-diffusion stochastic process. This is an example of hardcore driven lattice gas far from equilibrium that has proven to be an excellent system for theoretical investigations of low-dimensional systems out of thermal equilibrium. A large variety of phenomena had already been described by driven lattice gases (for an overview see [22–24] and the references therein). Using the rates given by Eqs. (1)–(6), one can rewrite the corresponding master equation as a Schrödinger-like equation in imaginary time,

$$\frac{\partial}{\partial t}|p(t)\rangle=-\mathcal{H}|p(t)\rangle. \tag{7}$$

The explicit form of $\mathcal{H}$ could be written down via the rate equations. Let $\langle n_{k,A}\rangle(\langle n_{k,B}\rangle)$ denote the probability that at time t, the site $N=k$ of the chain is occupied by a slow (fast) car. The Hamiltonian formulation of the master equation allows for evaluating the average quantities in a well-established manner. It could be easily verified that the following rate equations hold for the average occupation probabilities:

$$\frac{d}{dt}\langle n_{k,A}\rangle=h\langle n_{k-1,A}e_k\rangle+r\langle n_{k-1,B}e_k\rangle+\lambda\langle n_{k,B}n_{k+1,A}\rangle -h\langle n_{k,A}e_{k+1}\rangle-q\langle n_{k,A}e_{k+1}\rangle. \tag{8}$$

In the above equation, e_k stands for $1-n_{k,A}-n_{k,B}$. Similarly, for $\langle n_{k,B}\rangle$, we have

$$\begin{aligned}\frac{d}{dt}\langle n_{k,B}\rangle={}&q\langle n_{k-1,A}e_k\rangle+p\langle n_{k-1,B}e_k\rangle+s\langle n_{k-2,B}n_{k-1,A}e_k\rangle\\&-r\langle n_{k,B}e_{k+1}\rangle-p\langle n_{k,B}e_{k+1}\rangle-\lambda\langle n_{k,B}n_{k+1,A}\rangle\\&-s\langle n_{k,B}n_{k+1,A}e_{k+2}\rangle.\end{aligned} \tag{9}$$

Apparently, the total number of neither slow nor fast cars are conserved according to the dynamics and therefore the right-hand sides of Eqs. (8) and (9) cannot be written as a difference of two currents. However, the total number of cars, i.e., the sum of slow and fast cars, is a conserved quantity and the time rate of changing $\langle n_{A,k}\rangle+\langle n_{B,k}\rangle$ is equal to a difference between oncoming and outgoing currents. Summing up Eqs. (8) and (9) yields the following discrete form of the continuity equation:

$$\frac{d}{dt}[\langle n_{k,A}\rangle+\langle n_{k,B}\rangle]=\langle J_k^{in}\rangle-\langle J_k^{out}\rangle, \tag{10}$$

in which the explicit form of $\langle J_k^{out}\rangle$ is given below:

$$\langle J_k^{out}\rangle=h\langle n_{k,A}e_{k+1}\rangle+r\langle n_{k,B}e_{k+1}\rangle+q\langle n_{k,A}e_{k+1}\rangle +p\langle n_{k,B}e_{k+1}\rangle+s\langle n_{k,B}n_{k+1,A}e_{k+2}\rangle. \tag{11}$$

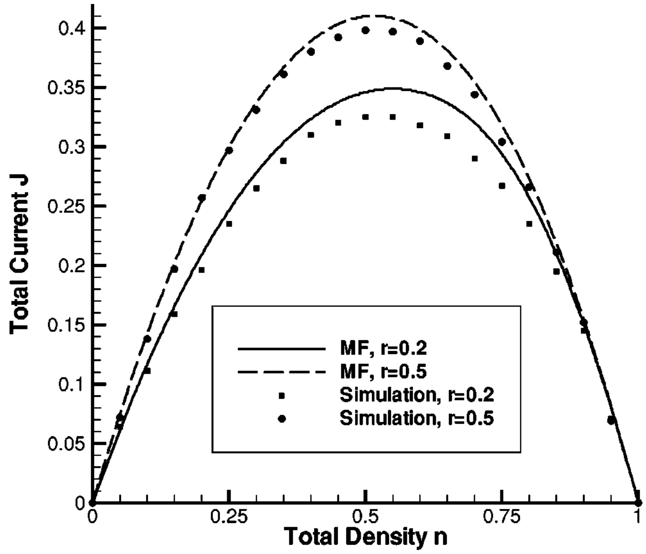


FIG. 1. Current-density diagram for different values of r. s is set to 0.4. The unit of current is number of cars passing each site per unit time of update.

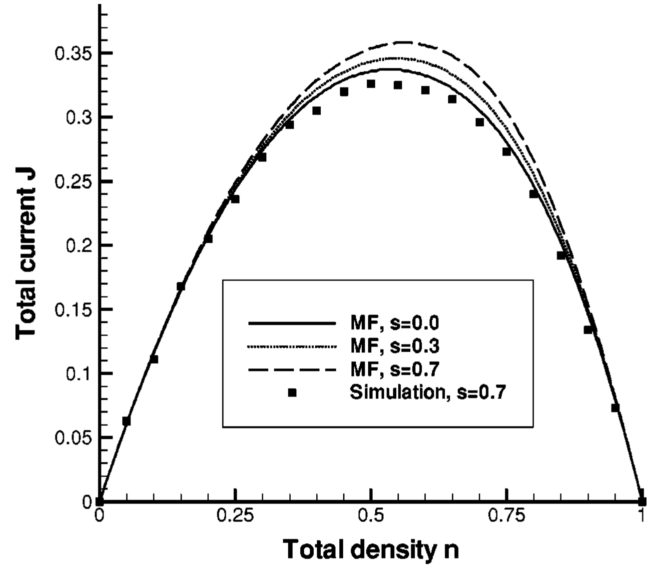


FIG. 2. Current-density diagrams for different values of s. r is set to 0.2. The current unit is the same as in Fig 1.

Equations (8), (9), and (11) are valid for arbitrary time t; however, our particular interest is focused on the long-time behavior of the system where stationarity is established. In the steady-state regime, one- and two-point correlators in Eqs. (8) and (9) will be time independent. Equation (10) implies that in steady state the current would be site independent as expected.

So far, our results have been exact and no approximation has been implemented. At this stage and in order to solve Eq. (8)–(11) we resort to a mean-field approximation and replace the two-point correlators with the product of one-point correlators. Moreover, since the closed boundary condition has been applied, it can be anticipated that the steady values of $\langle n_{k,A}\rangle_s$ and $\langle n_{k,B}\rangle_s$ be site independent and therefore we omit the site-dependence subscripts from Eqs. (8)–(11). Denoting the steady values of $\langle n_A\rangle_s$ and $\langle n_B\rangle_s$ by n_A and n_B, respectively, the steady current J turns out to be

$$J=(hn_A+rn_B+qn_A+pn_B+sn_An_B)(1-n). \qquad (12)$$

In the above expression, the total density of the cars has been taken to be n,

$$n_A+n_B=n. \qquad (13)$$

Our final aim is to write J in terms of total density n and the rates. This is performed if one writes n_A as a function n and the rates. By applying the mean-field approximation to Eq. (9) in its steady-state form, and using Eq. (13), one obtains the following equation:

$$r(n-n_A)(1-n)+\lambda(n-n_A)n_A=qn_A(1-n), \qquad (14)$$

which simply yields the solutions

$$n_A=\frac{1}{2\lambda}(n\lambda-(1-n)(q+r)\pm\{[n\lambda-(1-n)(q+r)]^2 +4rn(1-n)\lambda\}^{1/2}). \qquad (15)$$

The solution with the minus sign is unphysical ($n_A<0$) so the unique solution is the one with the positive sign. We remark that within the mean-field approach, one also can solve the time-dependent version of Eqs. (8) and (9). In this case, the equation for $\langle n_A\rangle$ turns out to be

$$\frac{d}{dt}\langle n_A\rangle=rn(1-n)-[(q+r)(1-n)-n\lambda]\langle n_A\rangle-\lambda(\langle n_A\rangle)^2, \qquad (16)$$

which simply give rises to the following solution:

$$\langle n_A\rangle(t)=\frac{n_A-C_1e^{C_2(C_3-t)}}{1-e^{C_2(C_3-t)}}, \qquad (17)$$

in which C_1, C_2, and C_3 are constants depending on the rates. In the long-time limit, the mean concentration of slow cars exponentially relaxes toward the steady value n_A. Re-

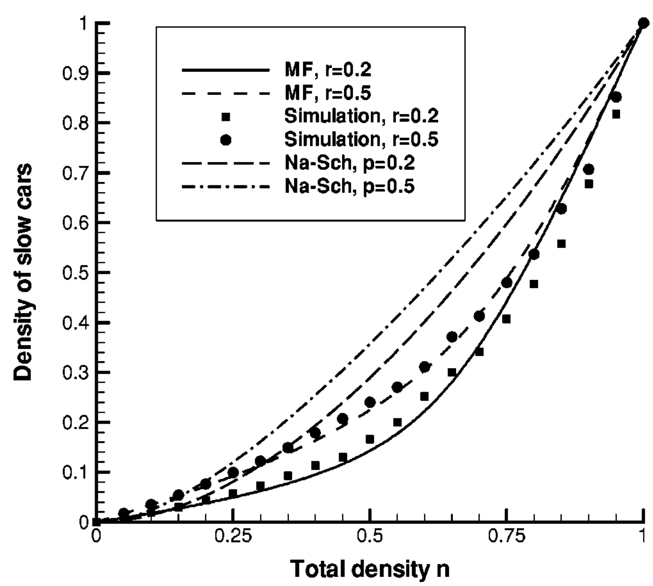


FIG. 3. Density of slow cars versus the total density for $s=0.4$.

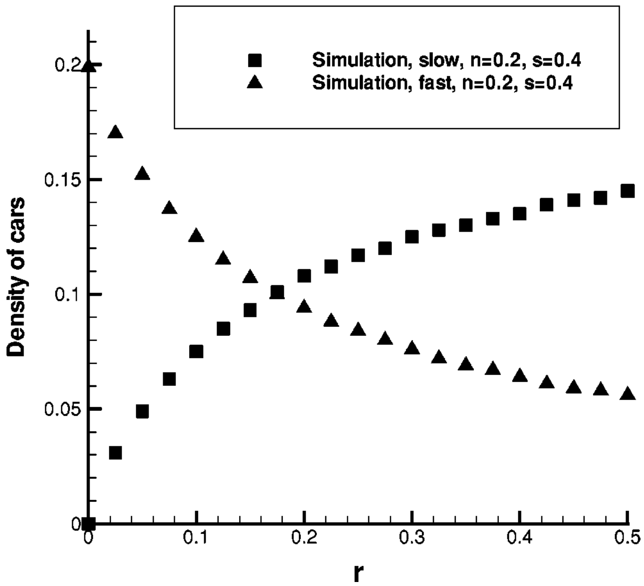


FIG. 4. Density of slow (fast) cars as a function of r. The value of n and s are 0.2 and 0.4, respectively.

placing the above n_A into Eq. (13), one now has the total current J as a function of n and the rates. In order to have better insights into the problem, extended computer simulations were carried out. Here we present the result of numerical investigations of model I. In these computer simulations, the system size is typically 2400. With no loss of generality, we rescale the time so that the rate of hopping a fast car is set to one. The speed of slow cars is supposed to be 70% of the speed of the fast cars, which is realized by taking $h=0.7$. The values of q and λ are set 1 and 0.7, respectively. One *subupdate* step consists of a random selection of a site, say, $N=i$, and developing the state of the link $(i,i+1)$ according to the dynamics. One *update* step contains L *subupdates*. The typical number of updates developed in order that the system reaches stationarity is 400 000 and the averaging has been performed over 500 000 updating steps. The initial state of the system was prepared randomly, i.e., each site is occupied with the probability n. Figures 1–6 show the result of numerical simulations.

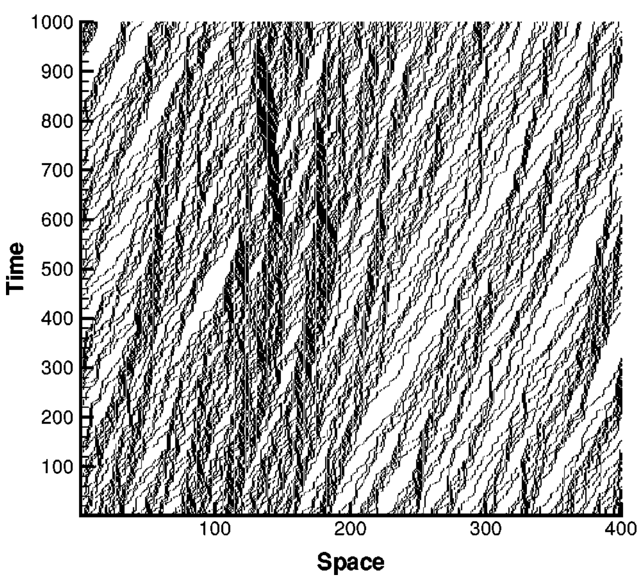
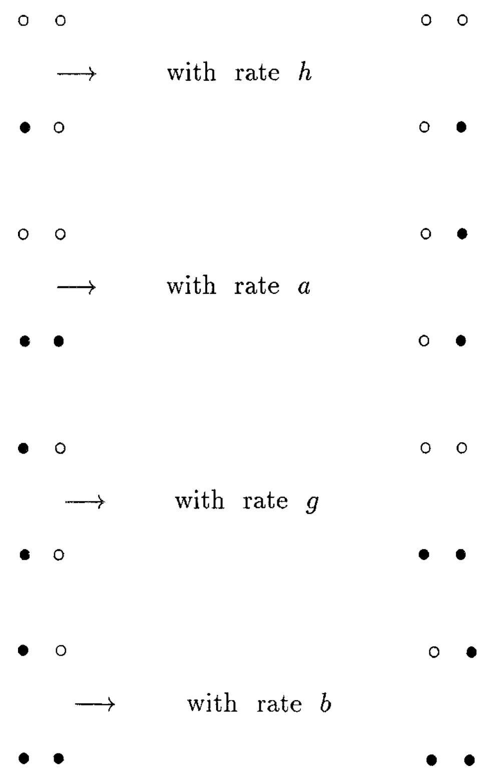


FIG. 5. Space-time diagram for $r=0.2$ and $s=0.0$. The unit of time is one update.

IV. MODEL II

A. Asymmetric regulation

The second model we consider has less resemblance to the NS model. Here, there is no specification of fast and slow cars and only one kind of particle exists in the chain; nevertheless the distinction between fast and slow cars is realized by their appearance in the passing and home lanes. In this periodic double-chain model, the following processes occur in a random sequential updating scheme:



As depicted, the *asymmetric* regulation has been adopted so that the top lane can only be used for passing. According to the above rules, once a successful passing has taken place, the passing car should return to its home lane unless the next site in the home lane is already occupied. In this circumstance, it can continue to pass the second slow car (multipassing). Each site of the double chain takes four different states, but according to the above dynamics only three of them appear in the course of time. The forbidden state is the one in which the passing-lane site is full and its parallel home-lane site is empty. Regarding this fact, we characterize the three allowed states by Φ,A, and B. Φ represents the situation where both parallel sites are empty, A represents the case of an occupied site in the home-lane and an empty parallel site in the passing lane, and finally, B refers to the case of both parallel sites being occupied.

This notation yields the following reaction-diffusion processes:

$$A\Phi \rightarrow \Phi A \quad (\text{with rate } h), \tag{18}$$

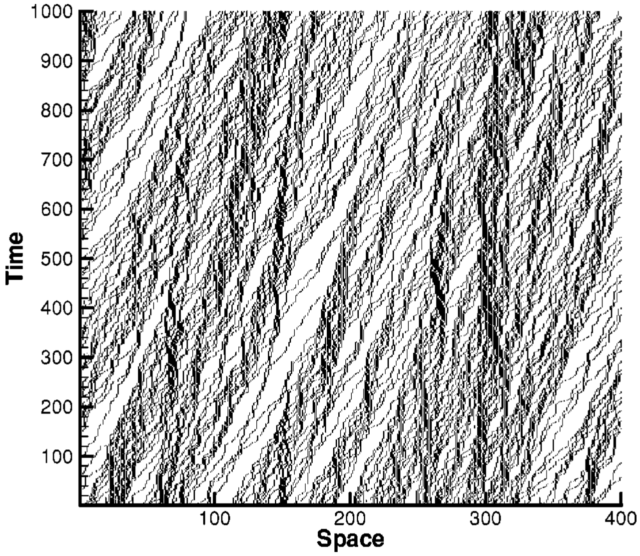


FIG. 6. Space-time diagram for $r=0.2$ and $s=0.7$. The unit of time is one update.

$$AA\to\Phi B \quad (\text{with rate } a), \tag{19}$$

$$B\Phi\to AA \quad (\text{with rate } g), \tag{20}$$

$$BA\to AB \quad (\text{with rate } b). \tag{21}$$

It is worth mentioning that the above model for a two-lane road is simultaneously being considered within the approach of deterministic cellular automata [25].

B. Master equation and mean-field approach

Similar to the steps performed in model I, one can write the following form of discrete-continuity equation:

$$\frac{d}{dt}[\langle n_{k,A}\rangle+2\langle n_{k,B}\rangle]=\langle J_k^{in}\rangle-\langle J_k^{out}\rangle \tag{22}$$

in which

$$\langle J_k^{out}\rangle=h\langle n_{k,A}e_{k+1}\rangle+b\langle n_{k,B}n_{k+1,A}\rangle+g\langle n_{k,B}e_{k+1}\rangle + a\langle n_{k,A}n_{k+1,A}\rangle. \tag{23}$$

The above expression for $\langle J_k\rangle$ has a clear interpretation in terms of rules (18)–(21). In steady state, the time dependences in the equation disappear and the current will be site independent. Next, we apply the mean-field approximation through which all the two-point correlators are replaced by the product of one-point correlators. This leads to the following equation for J:

$$J=hn_A(1-n)+b\left(n-\frac{n_A}{2}\right)n_A+g\left(n-\frac{n_A}{2}\right)(1-n)+an_A^2, \tag{24}$$

where the relation $n_A/2+n_B=n$ has been used.

In order to obtain J in terms of total density n and the rates, we must write n_A as a function of n and the rates. This is done by solving the following equation with its left-hand side set to zero.

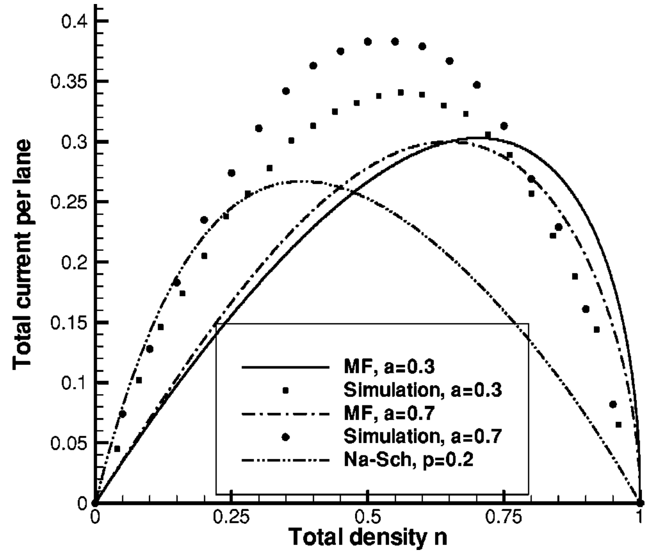


FIG. 7. Current-density diagram for different values of passing rates. The current unit is the same as in Fig. 1.

$$\frac{d}{dt}n_A=2gn_B(1-n)-2an_A^2. \tag{25}$$

The unique physical solution of the above equation is

$$n_A=\frac{1}{4a}(\{[g^2(1-n)^2+16an(1-n)g]\}^{1/2}-g(1-n)). \tag{26}$$

Putting Eq. (26) in Eq. (24), the current J is now obtained in terms of n and the rates. The result of computer simulations are shown in Figs. 7–9. Here the rates b, g, and h are chosen to be 1.0, 1.0, and 0.7, respectively, while a is varied. We recall that a measures the tendency of fast cars to pass the slow ones. The simulation specifications are the same as those in model I.

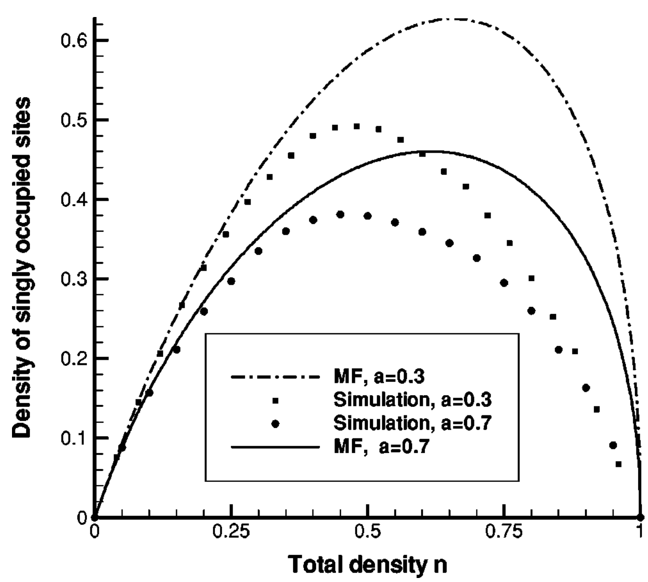


FIG. 8. Density of singly occupied sites versus the total density.

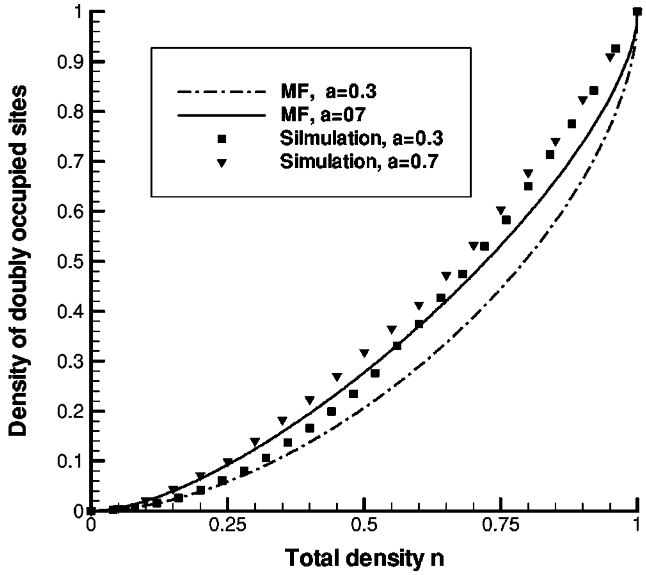


FIG. 9. Density of doubly occupied sites versus the total density.

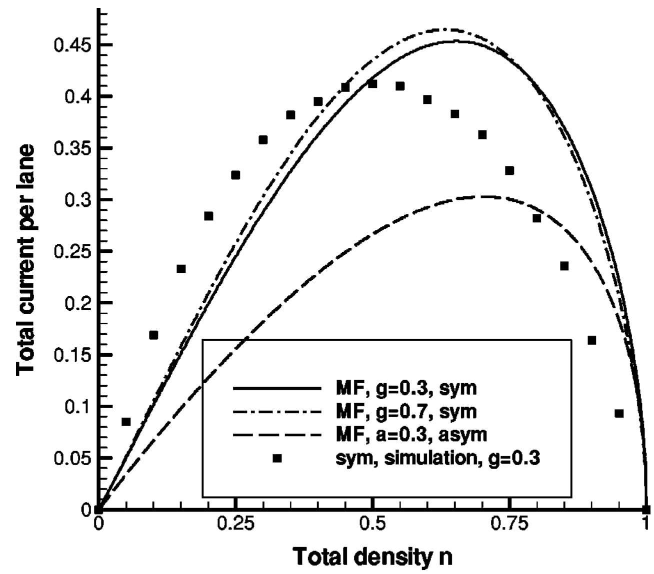


FIG. 10. Current per lane-density diagram for different values of passing rates.

C. Symmetric regulation

Here we allow the fast cars to pass rightward as well. In this case, both the top and bottom lanes become identical and fast cars can pass the slow ones irrespective of their home lane. In this *symmetric* two-lane model, each particle hops one site ahead in its home lane, provided that the next site is empty. Otherwise it tries to pass the car ahead. This attempt is successful if there is an empty site ahead on the opposite lane. The following rules illustrates the model definition:

$$\begin{matrix} * & * \\ \bullet & \circ \end{matrix} \xrightarrow{\text{with rate } h} \begin{matrix} * & * \\ \circ & \bullet \end{matrix}$$

$$\begin{matrix} \bullet & \circ \\ * & * \end{matrix} \xrightarrow{\text{with rate } h} \begin{matrix} \circ & \bullet \\ * & * \end{matrix}$$

$$\begin{matrix} \bullet & \bullet \\ \circ & \circ \end{matrix} \xrightarrow{\text{with rate } g} \begin{matrix} \circ & \bullet \\ \circ & \bullet \end{matrix}$$

$$\begin{matrix} \circ & \circ \\ \bullet & \bullet \end{matrix} \xrightarrow{\text{with rate } g} \begin{matrix} \circ & \bullet \\ \circ & \bullet \end{matrix}$$

The asterisk symbols indicate that the process in the opposite lane occurs independently of the configuration of the sites filled with an asterisk. If we denote the state of two parallel sites in which the bottom site is empty and the top one is occupied by B, the state of simultaneous occupation of parallel sites by C and adopting the notations Φ and A as the same in the asymmetric version of the model, then it could easily be verified that the forms of the discrete-continuity equation and the current one are as follows:

$$\frac{d}{dt}(\langle a_k\rangle+\langle b_k\rangle+2\langle c_k\rangle)=\langle J_{k-1}\rangle-\langle J_k\rangle \tag{27}$$

and

$$\begin{aligned}\langle J_{k,k+1}\rangle={}&h(\langle a_k e_{k+1}\rangle+\langle a_k b_{k+1}\rangle+2\langle c_k e_{k+1}\rangle+\langle c_k b_{k+1}\rangle\\&+\langle b_k e_{k+1}\rangle+\langle b_k a_{k+1}\rangle+\langle c_k a_{k+1}\rangle)+g(\langle b_k b_{k+1}\rangle\\&+\langle a_k a_{k+1}\rangle),\end{aligned} \tag{28}$$

where $\langle a_k\rangle$, $\langle b_k\rangle$, and $\langle c_k\rangle$ refer to the probabilities that at time t, the site $N=k$ of the double chain has one car in the

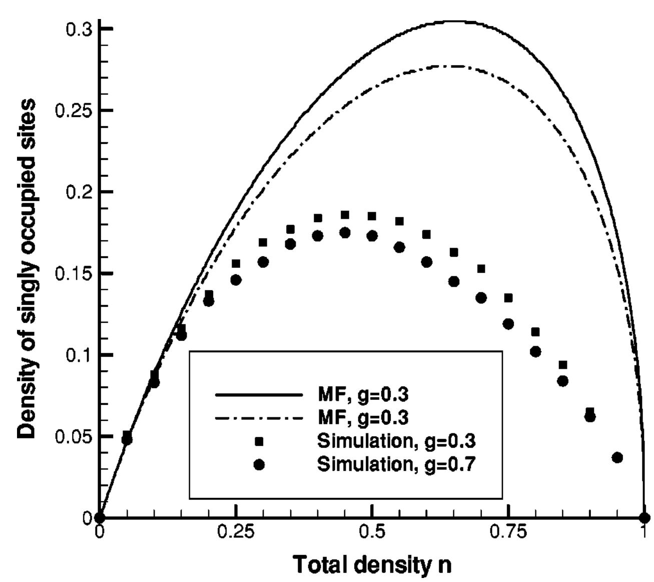


FIG. 11. Density of singly occupied sites versus the total density.

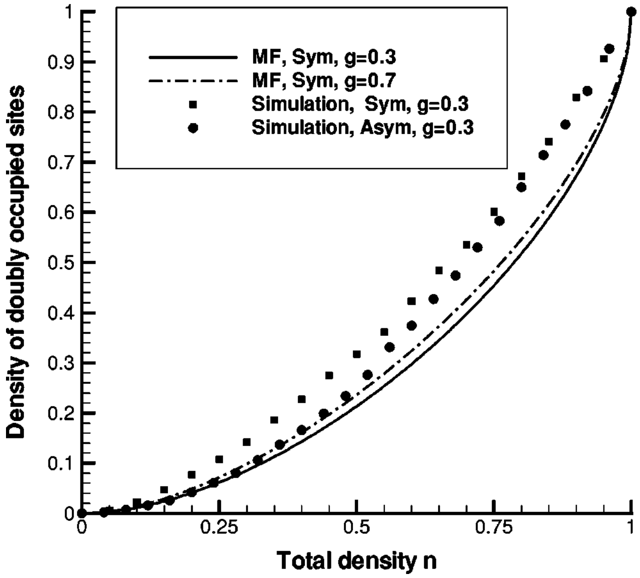


FIG. 12. Density of doubly occupied sites versus the total density.

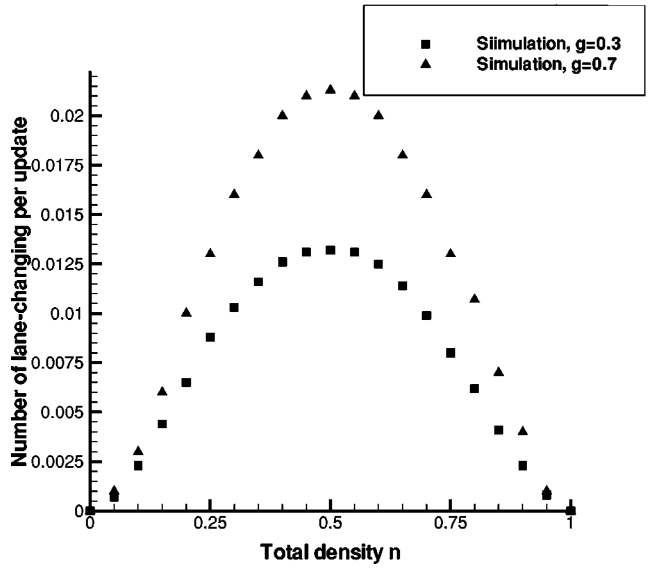


FIG. 13. Number of lane-changing per update versus total density.

bottom lane, one car in the top lane, and double occupancy in both lanes, respectively. In the steady state, the system is both time and site independent. Denoting the steady values of $\langle a_k \rangle$, $\langle b_k \rangle$, and $\langle c_k \rangle$ by a, b, and c, one has the relation

$$\frac{a+b}{2}+c=n. \tag{29}$$

Moreover, the symmetry between the lanes implies that $a=b$. The steady value a is easily found to be obtained from the following equation:

$$(g+h)a^2=hc(1-n). \tag{30}$$

Solving the steady-state equation for a, one finds

$$a=\frac{\{[h^2(1-n)^2+4hn(1-n)(g+h)]\}^{1/2}-h(1-n)}{2(g+h)}. \tag{31}$$

Also, Eq. (31) leads to the following equation for J:

$$J=2[hn(1-n)+h\{a^2+2a(n-a)\}+ga^2], \tag{32}$$

where, by putting Eq. (31) into it, one reaches the expression for J in terms of n, g, and h. We remark that the factor 2 reflects the number of lanes. The result of computer simulations are shown in Figs. 10–13 The value of h is set to one and g is varied.

V. CONCLUDING REMARKS

We have introduced a two-species reaction-diffusion model for description of a unidirectional two-lane road. The type of update we have used is random sequential, which sounds more appropriate for analytical treatments. In the first model, the results of numeric simulations are very close to those in the mean-field approach, which indicates that the effects of correlations are suppressed. However, in the second model, there are remarkable differences between analytical and numeric results. In model I, the current-density diagram is slightly affected by changing the passing rate and the passing process has most effect in the intermediate densities. This could be anticipated since in the low and high densities, the number of passing considerably reduces. The space-time diagrams of the model I reveal the discriminating effect of passing.

In model II (both symmetric and asymmetric), the maximum of J occurs in different values of n in simulation and in the analytical approach. The mean field predicts a shift toward higher densities, while in simulation a slight shift toward the left is observed. We note that in the PCA-based models, the maximum of J corresponds to a considerable left-shifted value of the density [16,17]. In the symmetric version of the model II, we observe an increment of the current with regard to the asymmetric version. In contrast to the asymmetric version, the maximum of J in the mean-field approach is higher than its value obtained through simulation. Although the current diagram (10) appears asymmetrically with respect to the density, the lane-changing diagram (13) is symmetric to a high accuracy.

ACKNOWLEDGMENTS

I would like to express my gratitude to V. Karimipour and G. Schütz for their fruitful comments. I acknowledge the assistance given by A. Schadschneider, N. Hamadani, V. Shahrezaei, and R. Sorfleet and in particular I highly appreciate R. Gerami for his valuable help in the computer simulations.

[1] D. Chowdhury, L. Santen, and A. Schadschneider, Curr. Sci. **77**, 411 (1999); Phys. Rep. **329**, 199 (2000).
[2] *Traffic and Granular Flow*, edited by D. E. Wolf, M. Schreckenberg, and A. Bachem (World Scientific, Singapore, 1996).
[3] *Traffic and Granular Flow*, edited by D. E Wolf and M. Schreckenberg (Springer, Singapore, 1998).
[4] *Traffic and Granular Flow*, edited by H. J. Herrmann, D. Helbing, M. Schreckenberg, and D. E. Wolf (Springer, Berlin, 2000).
[5] D. Helbing, *Verkehrsdynamik* (Springer, Berlin, 1998).
[6] L.A. Pipes, J. Appl. Phys. **24**, 274 (1953).
[7] M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama, Jpn. J. Indust. Appl. Math, **11**, 203 (1994).
[8] B.S. Kerner, S.L. Klenov, and P. Konhaüser, Phys. Rev. E **56**, 4200 (1997).
[9] H.Y. Lee, H.-W Lee, and D. Kim, Phys. Rev. E **59**, 5101 (1999).
[10] I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic* (Elsevier, Amsterdam, 1971).
[11] D. Helbing and M. Treiber, Science **282**, 2002 (1998).
[12] M. Schreckenberg, A. Schadschneider, K. Nagel, and N. Ito, Phys. Rev. E **51**, 2939 (1995).
[13] A. Schadshneider, Eur. Phys. J. B **10**, 573 (1999).
[14] L. Santen, Dissertation, Universität zu Köln, 1999 (unpublished).
[15] K. Nagel and M. Schreckenberg, J. Phys. I **2**, 2221 (1992).
[16] M. Rickert, K. Nagel, M. Schreckenberg, and A. Latour, Physica A **231**, 534 (1996).
[17] K. Nagel, D.E. Wolf, P. Wagner, and P. Simon, Phys. Rev. E **58**, 1425 (1998).
[18] P.M. Simon and H.A. Gutowitz, Phys. Rev. E **57**, 2441 (1998).
[19] K. Nagel, J. Esser, and M. Rickert, Annu. Rev. Comput. Phys. **VIII** (2000).
[20] D. Chowdhury and A. Schadschneider, Phys. Rev. E **59**, R1311 (1999).
[21] T. Nagatini, J. Phys. A **29**, 6531 (1996).
[22] G. M. Schütz, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic Press, London, 2000).
[23] *Non-Equilibrium Statistical Mechanics in One Dimension*, edited by V. Privman (Cambridge University Press, Cambridge, England, 1997).
[24] F.C. Alcaraz, M. Droz, M. Henkel, and V. Rittenberg, Ann. Phys. (N.Y.) **230**, 250 (1994).
[25] V. Belitsky, J. Krug, E. Jordao, and G. M. Schütz, J. Stat. Phys. (to be published).